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ORIGINAL ARTICLE

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Separation vortices and pattern formation

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Abstract In this paper examples are given of the importance of flow separation for fluid patterns at moderate Reynolds numbers—both in the stationary and in the time-dependent domain. In the case of circular hydraulic jumps, it has been shown recently that it is possible to generalise the Prandtl–Kármán–Pohlhausen approach to stationary boundary layers with free surfaces going through separation, and thus obtain a quantitative theory of the simplest type of hydraulic jump, where a single separation vortex is present outside the jump. A second type of jump, where an additional roller appears at the surface, cannot be captured by this approach and has not been given an adequate theoretical description. Such a model is needed to describe "polygonal" hydraulic jumps, which occur by spontaneous symmetry breaking of the latter state. Time-dependent separation is of importance in the formation of sand ripples under oscillatory flow, where the separation vortices become very strong. In this case no simple theory exists for the determination of the location and strengths of separation vortices over a wavy bottom of arbitrary profile. We have, however, recently suggested an amplitude equation describing the long-time evolution of the sand ripple pattern, which has the surprising features that it breaks the local sand conservation and has long-range interaction, features that can be underpinned by experiments. Very similar vortex dynamics takes place around oscillating structures such as wings and fins. Here, we present results for the vortex patterns behind a flapping foil in a flowing soap film, which shows the interaction and competition between the vortices shed from the round leading edge (like the von Kármán vortex street) and those created at the sharp trailing edge.

Keywords Separation · Vortex formation · Hydraulic jump · Sand ripples · Vortex street

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1 Introduction

Flow separation is at the basis of many interesting structures at Reynolds numbers, where both inertia and viscosity are important. In this paper we shall give examples of such flows as they occur in hydraulic jumps, sand ripples and wakes—areas where progress has been made recently although there still is a strong need of better understanding. Separation typically occurs in flows past a sharp edge or rapid flows along a solid boundary, when the friction becomes so large that part of the flow starts moving backwards. When separation develops, it typically occurs in a finite region of the flow, called a separation vortex—or bubble. The theory of separation is notoriously difficult. The reason is that separation occurs in connection with boundary layers

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and the standard boundary layer theory due to Prandtl becomes singular at separation points. Since most structurally interesting problems occur at moderate Reynolds number, scaling approaches (such as "Triple deck" expansions designed for $Re \to \infty$) do not work very well.

2 Circular and polygonal hydraulic jumps

In a circular hydraulic jump, a jet of fluid impinges on a flat solid surface. As shown in Fig. 1a, the rapid flow in the thin layer (thickness around 1 mm) close to the jet switches suddenly to a slow, thick layer (thickness around 1 cm) at a well-defined, circular boundary—the circular hydraulic jump. This is an example of stationary separation [1–3] as shown in Fig. 1b. The standard modelling of such flows is made by shallow water theory in which the jump appears as a shock between supercritical (rapid) and subcritical (slow) flows [4]. It has, however, recently been shown [5] that it is possible to compute the flow fairly accurately by a simple system of ODEs derived from boundary layer theory, modelling the radial velocity profile $u(r, z)$ as a cubic polynomial, allowing a separated profile. Thus, for a flow with height profile $h(r)$ we assume that $u(r, z)/v(r) = a(r)\eta + b(r)\eta^2 + c(r)\eta^3$ where $\eta = z/h(r)$ and $v(r)$ is the average velocity through the layer. Due to the boundary conditions (kinematic and dynamic) on the free surface, the coefficients a, b, and c can be expressed in terms of a single shape parameter $\lambda(r)$ as, for example $a = \lambda + 3$, $b = -(5\lambda + 3)/2$, and $c = 4\lambda/3$. The separation condition is $\partial u(r^*, z)/\partial z = 0$ at the bottom ($z = 0$), i.e., $a = 0$, or $\lambda = -3$.

The dynamics is governed by the boundary layer equations and to obtain a closed system of ODEs with the above ansatz we need to satisfy two moments. In [5] they were chosen as the average of the momentum equation through the layer and the limit of the momentum equation on the bottom $z = 0$. We then obtain two coupled non-autonomous ODEs for h and λ of the form

$$h' = -\frac{5\lambda + 3}{rh^3}; \quad G'(\lambda)\lambda' = \frac{4r\lambda}{h} + G(\lambda)\frac{h^4 - (5\lambda + 3)}{rh^4} \tag{1}$$

where

$$G(\lambda) = \frac{1}{h}\int_0^h \left(\frac{u}{v}\right)^2 \mathrm{d}z = \frac{6}{5} - \frac{\lambda}{15} + \frac{\lambda^2}{105}. \tag{2}$$

Here, the flux $q = rh(r)v(r)$ has been normalised to unity by re-scaling the variables. This model does become singular, but only on the lines $h = 0$ and $\lambda = 7/2$. It is therefore possible to describe a flow with separation ($\lambda < -3$). Similar approaches are possible for hydraulic jumps in channels [6].

The type of jump described so far typically occurs when the jet is impinging on a (not too large) flat plate and simply allowed to exit by falling over the rim. If the outflow of the fluid is partly blocked by placing a weir at the rim an interesting transition is observed as the weir height, and thereby the external fluid height h_{ext}, is increased [7]. Crossing a threshold of h_{ext}, the jump becomes unstable and acquires a new structure with a larger jump region between two concentric circular loci, and in this jump region the surface flow is *backwards*.

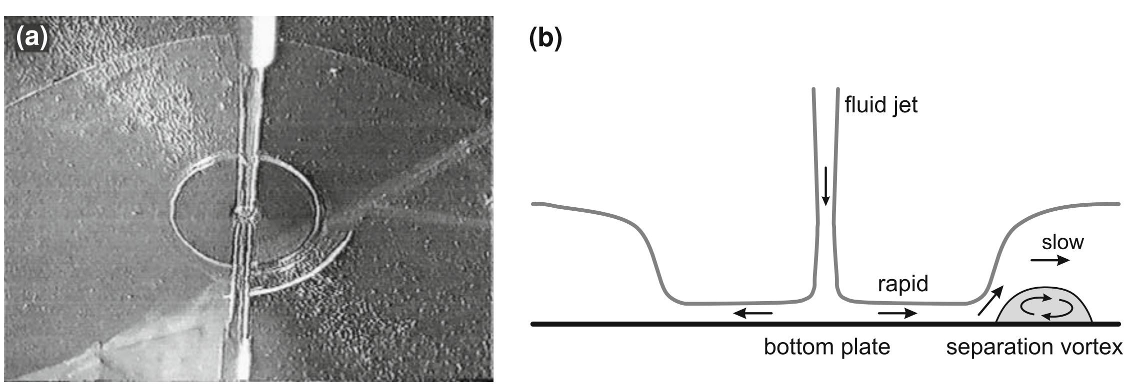

Fig. 1 **a** Circular hydraulic jump in a "kitchen sink". **b** Schematic drawing of the flow in a circular hydraulic jump. The *grey region* is a separation vortex

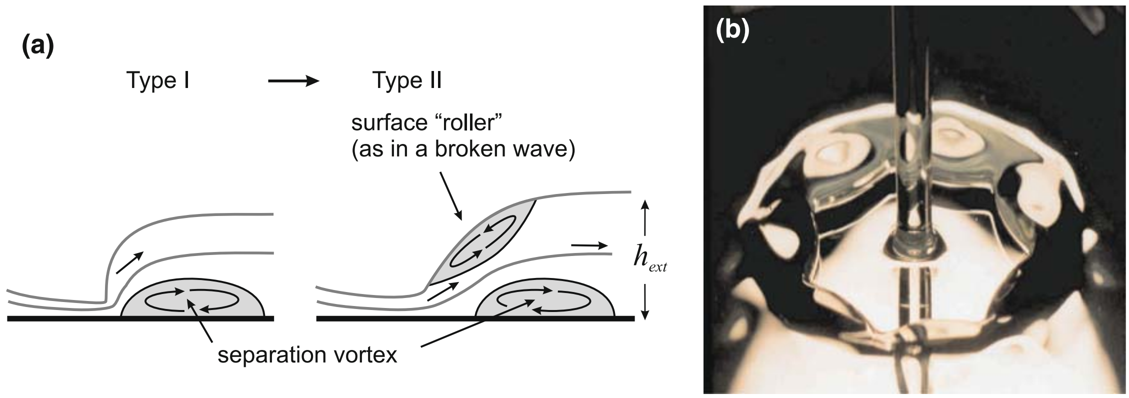

Fig. 2 **a** Schematic drawing contrasting the flows in type I and type II circular hydraulic jumps. **b** A heptagonal hydraulic jump with ethylene glycol. Courtesy of Clive Ellegaard

Thus a “roller”, i.e., a separation vortex at the surface has now appeared as sketched in Fig. 2a. This type II state is typically unstable and the circular symmetry is spontaneously broken [8], leading, for sufficiently viscous fluids with sufficient surface tension [9], to a stationary polygonal structure as shown in Fig. 2b. Close to the transition there can be many corners (up to 13 has been observed) but as h_{ext} is increased further, the number of corners gradually diminish, although several polygonal structures can be stable simultaneously. The outer circle remains and is still marked by a (weak) separation vortex on the bottom. No theoretical description exists for the type II hydraulic jump—even as regards the circular state, and thus only phenomenological models exist for the polygons.

3 Sand ripples under oscillatory flow

Another case in which separation—in this case *time dependent*—plays a major role, is the formation and dynamics of the sand ripples that form under oscillatory flow. Such structures are readily observed in the shallow water near the beach and the pioneering works on this type of ripples were made by Ayrton [10], Bagnold and Taylor [11] many years ago. From their work as well as many more recent (e.g., [12–14]), the key role played by separation vortices is evident. This implies also that the wave length selection is simple: like the maximal size of the separation vortex, the wavelength is roughly proportional to the amplitude of the horizontal water displacement.

The full description of sand ripples is very complicated. The flow is turbulent and the precise modelling of the granular transport is thus very demanding. But still, the most important overall features would be predictable if a general theory of separation of laminar flow along a wavy bottom was known. Short of this, we have recently succeeded in constructing an amplitude equation, which can describe many of the pertinent long-time features of the dynamics, without actually solving for the separation vortex pattern [15]. The amplitude equation describes the sand ripples entirely in terms of the local height h (the elevation of the sand-bottom). It does not resolve the periodic drive, but only time scales much longer than this period [16]. The flat bed is unstable [17], but a sufficient inclination (i.e., one larger than the angle of repose) will quench the instability. The lowest order equation with these properties has the structure

$$h_t = -\epsilon(h - \bar{h}) + \left((h_x)^2 - 1\right) h_{xx} - h_{xxxx} + \delta \left((h_x)^2\right)_{xx} \qquad (3)$$

where $\bar{h}$ is the average height over the one-dimensional spatial variable x, and ϵ and δ are coefficients depending on external conditions. The first term (proportional to ϵ) in this equation is unexpected and lack of appreciation of this has delayed the development of an appropriate amplitude equation [18]. It violates two basic principles, which are usually taken for granted: local sand conservation and locality. Sand must of course be locally conserved in the bulk and one might thus off-hand assume that the equation should have the form $h_t = -q_x$ with some appropriate flux q. This presumes, however, that sand is only transported along the surface of the ripples. In fact it is known that sand is transported both into the bulk [19] and carried by the flow, and therefore this assumption is not necessarily true. Even including the ϵ-term, sand is globally conserved (with periodic boundary conditions), i.e., the inclusion of the average height $\bar{h}$ explicitly into the equation ensures that this

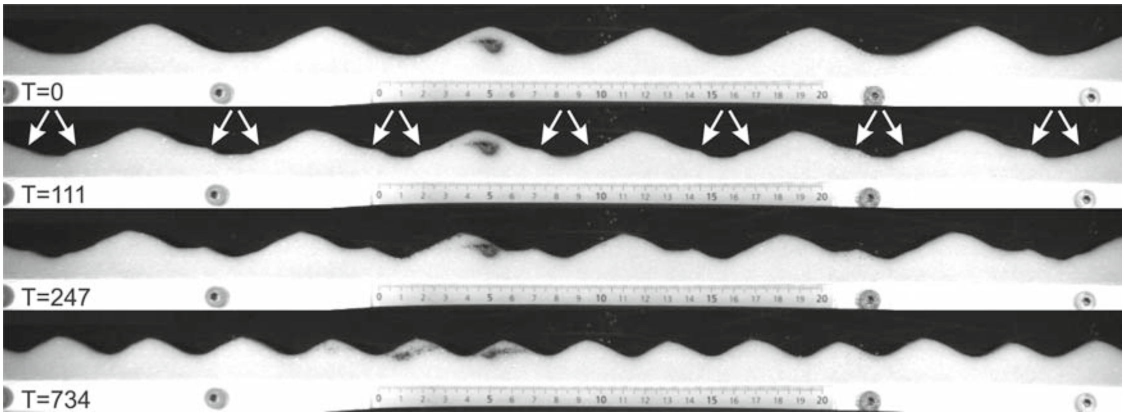

Fig. 3 Dynamics of the doubling transition. As shown by the *arrows*, two new satellite ripples are initially formed in each trough. After a short time one satellite wins (here the left one) and this choice is obeyed in all the troughs. T is the time in oscillation periods. The length scale in centimetres can be seen from the ruler

quantity is independent of time. But this means that the growth of an individual ripple depends on the average ripple height—a global property. In a sufficiently large system, this would not be possible, but, as we shall see below, the hydrodynamical interactions are surprisingly long-ranged.

The initial instability from a flat bed occurs in the model around the most unstable wavenumber $k = 1/\sqrt{2}$ and subsequently the wavelength increases by coarsening. For $\epsilon = 0$ the coarsening would proceed without bound, until only one giant ripple was present [18,20,21]. For $\epsilon > 0$, however, the process stops at a finite wavelength as in the experiments. When the amplitude or the frequency of the tray is suddenly changed, a well-defined set of bifurcations typically take place [22,23]. Increasing the frequency (which does not affect the wavelength much) leads to a "pearling" transition, where a checkerboard of small ripples ("pearls") appear in the troughs of the old ones in a reversible transition. Increasing the amplitude above a threshold value leads to coarsening in the form of "bulging" where the ripples undergo varicose undulations leading to dilations and compressions and subsequent disappearance of ripples at compression points. Finally, a sufficient decrease in amplitude introduces new ripples in the system. But this can apparently only happen by initially doubling the number of ripples by placing a new one in each trough—although the number of ripples thereby becomes too large. This transition has been called "doubling", and a new ripple emerges in each trough. In the final stages superfluous ripples are subsequently removed by "bulging". Of these transitions, only the doubling transition can be realised in a quasi one-dimensional system (a narrow channel). For this to occur in our amplitude Equation (3), it is crucial that $\delta > 0$, since $\delta = 0$ leads to up-down symmetric ripples where creation of new ripples cannot take place exclusively in the troughs.

The short-time dynamics of the doubling transition reveals an interesting detail, showing the long-range character of the interaction. Figure 3 shows a sequence of height profiles from an experimental doubling transition. The initial instability creates two new satellite ripples in each trough, i.e., a kind of ripple "tripling". Only one of these satellites, however, survives. This can be the left or the right satellite, but, surprisingly, the same satellite wins in all troughs. We observe both left and right winners, so we believe that this selection is not due to lack of symmetry in our experiment, but is due to spontaneous symmetry breaking. The effect is so strong that if we stop the system and move one of the ripples to the other side of the trough, the system will pull it back in line when it is restarted. Our simple amplitude equation does not capture this effect, and indeed we do not expect Equation (3) to be accurate on time scales of the order of the driving period. But it does show that the ripples are strongly coupled and that a more refined theory is needed, which includes the time-dependent separation vortices explicitly.

4 Wakes in a soap film

Vortices can be beautifully visualised in soap films and thus dynamic separation processes can be explored effectively in soap films flowing around solid bodies [24–26]. In Fig. 4 we show visualisations of vortices behind a flapping foil in a vertically flowing soap film. By varying frequency and amplitude of the periodic oscillation we obtain a variety of wakes of relevance for fish swimming and with great aesthetic appeal. The

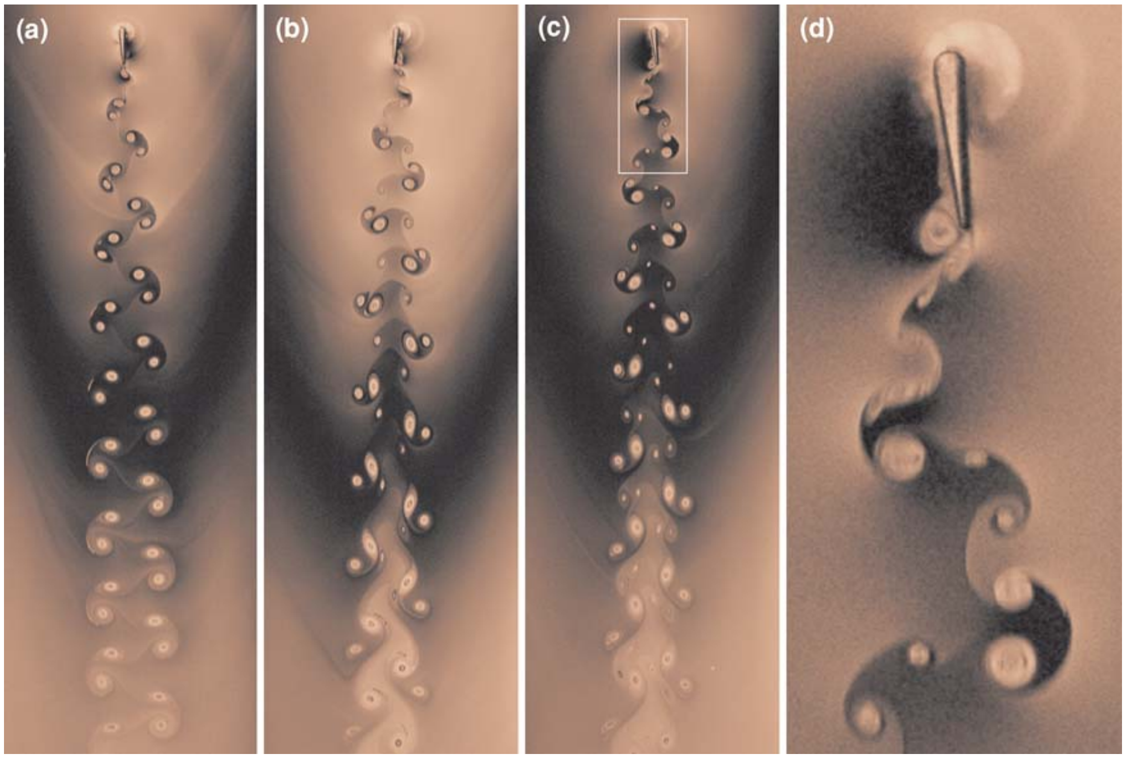

Fig. 4 Wakes generated by a flapping foil in a flowing soap film. **a** 2P wake at $St_D = 0.085$, $A_D = 1.2$. **b** 2P+2S wake at $St_D = 0.074$, $A_D = 1.5$. **c** 4P wake at $St_D = 0.069$, $A_D = 1.6$. The wake notation follows [27] where "S" signifies a single vortex and "P" signifies a pair of vortices of opposite signs. **d** Magnification of the area highlighted in (**c**) showing clearly the vortex formation process while the foil moves from left to right

vorticity in the wake is correlated to thickness variations of the film, which we visualise using a sodium lamp to produce interference patterns. Our experimental setup allows us to study both the vortex shedding at the trailing edge and the wake evolution far downstream. The foil is 1 mm wide and 6 mm long, it has a round leading edge and a sharp trailing edge, and it is driven with symmetric pitching oscillations about the 1/12 chord point. The experiment is characterised by three dimensionless parameters, i.e., the Strouhal number $St_D = Df/U$, the dimensionless amplitude $A_D = 2A/D$, and the Reynolds number $Re = DU/\nu$, where D is the width of the foil, f is the oscillation frequency, U is the free-stream speed, A is the flapping amplitude, and ν is the kinematic viscosity of the soap film.

In the experiment, we vary St_D and A_D while keeping Re constant at approximately $2 \cdot 10^2$ and in this way we can obtain a number of vortex wake patterns behind the foil, including normal (von Kármán) and inverted (von Kàrmàn) vortex streets. As has been noted earlier for a cylinder oscillated transversely to a free stream [27], the shedding frequency is locked to the imposed frequency f. The "leading edge" vortices generated at the rounded upstream edge of the foil travel down-stream and interact at the tip with the vortices formed at the trailing edge due to the flapping motion. As shown in Fig. 4 the resulting patterns can be rather complicated with several vortices or pairs forming in each cycle. Patterns analogous to Fig. 4a have also been visualised in water tunnels [28,29]. These patterns are of importance for swimming and flying animals, which move by flapping their fins or wings. In particular, we would like to understand the thrust generated and for that purpose a simple model of the vortex generation would be important—analogous to other separating flows.

5 Conclusion

We have shown that separation plays a large role in the formation of flow structure in many systems, either with free surfaces, with interfaces or simply near solid bodies where boundary layers can form. It is a challenge to develop theories that can account for separation under quite general conditions, e.g., along a wavy boundary or near a free surface—and in both stationary and time-dependent settings.

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