

Low-power optical communication: Approaching the quantum limit

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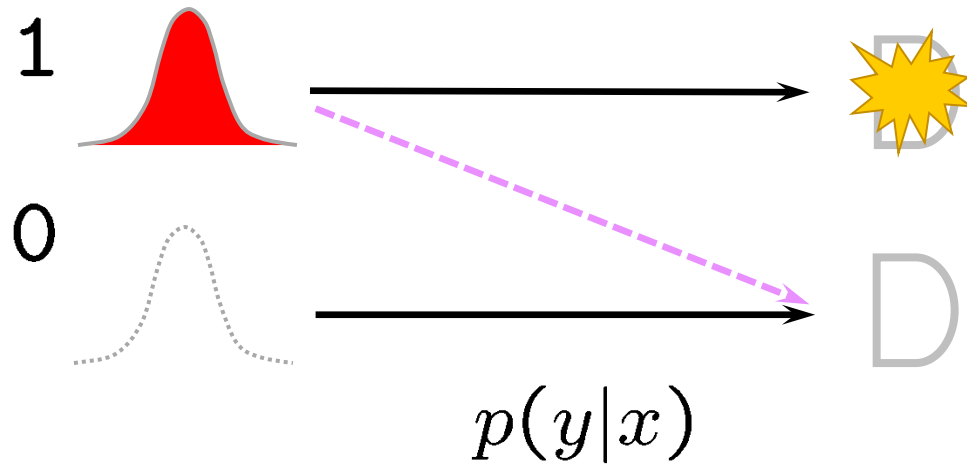
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OOK: On-off keying

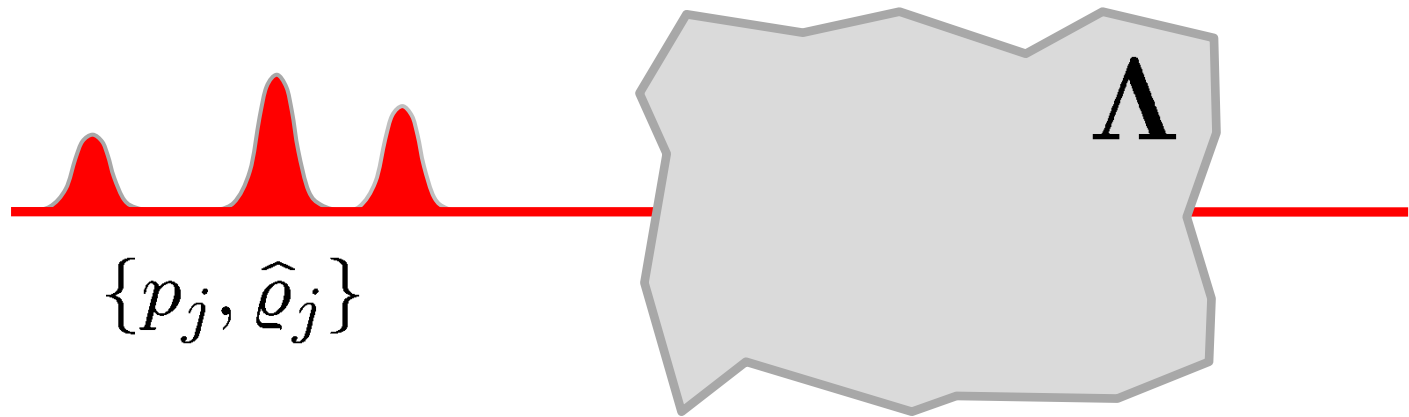
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Mutual information:

$$I(X, Y) = \sum_{x,y} p(y|x)p(x) \log_2 \frac{p(y|x)}{\sum_{x'} p(y|x')p(x')}$$

Channel capacity



Holevo quantity

$$\chi(\{p_j, \hat{q}_j\}) = S\left(\sum_j p_j \Lambda(\hat{q}_j)\right) - \sum_j p_j S(\Lambda(\hat{q}_j))$$

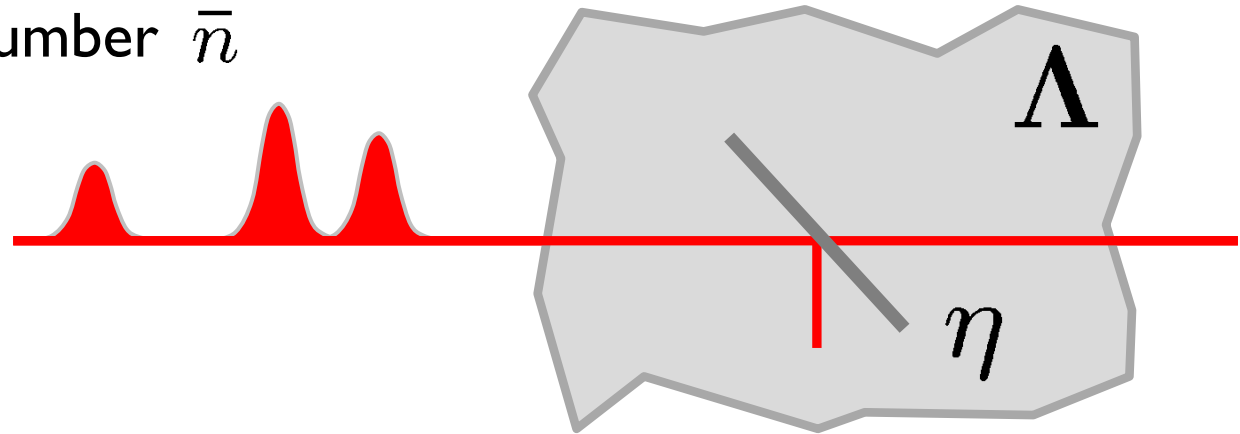
Classical channel capacity

$$C = \max_{\{p_j, \hat{q}_j\}} \chi(\{p_j, \hat{q}_j\})$$

Lossy bosonic channel

V. Giovannetti, R. García-Patrón, N. J. Cerf, A. S. Holevo,
Nature Photon. **8**, 796 (2014)

Average input
photon number \bar{n}



Channel capacity

$$C = (\eta\bar{n} + 1) \log_2(\eta\bar{n} + 1) - \eta\bar{n} \log_2 \eta\bar{n}$$

\equiv entropy of a thermal state with the mean output
photon number $\eta\bar{n}$

$$\approx \eta\bar{n} \log_2 \frac{1}{\eta\bar{n}} + O(\eta\bar{n}) \quad \text{for } \eta\bar{n} \ll 1$$

Optimal ensemble: lossless case

Fock states ($n = 0, 1, 2, \dots$):

$$p_n = \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n, \quad \hat{\rho}_n = |n\rangle \langle n|$$

Coherent state ensemble (α arbitrary complex):

$$p_\alpha = \frac{1}{\pi \bar{n}} e^{-|\alpha|^2 / \bar{n}}, \quad \hat{\rho}_\alpha = |\alpha\rangle \langle \alpha|$$

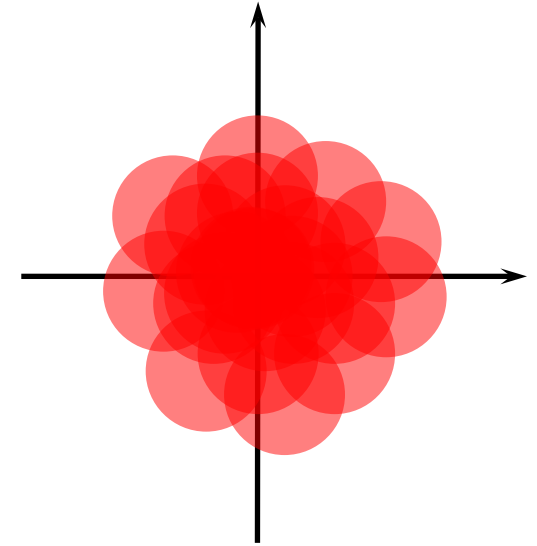
Average state:

$$\hat{\rho}_{\text{av}} = \sum_{n=0}^{\infty} p_n \hat{\rho}_n = \int_{\mathbb{C}} d^2\alpha p_\alpha \hat{\rho}_\alpha$$

Losses: Coherent state ensemble

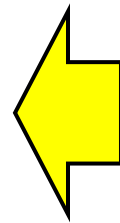
Coherent states remain *pure*:

$$\Lambda(\hat{\rho}_\alpha) = |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha|$$



Average output state

$$\int d^2\alpha p_\alpha \Lambda(\hat{\rho}_\alpha)$$



Thermal state
with $\eta\bar{n}$ photons
on average

- Continuum of input states
- Phase reference needed

Losses: Fock state ensemble

$$\Lambda(\hat{\rho}_n) = \sum_{k=0}^n \binom{n}{k} \eta^k (1 - \eta)^{n-k} |k\rangle \langle k|$$

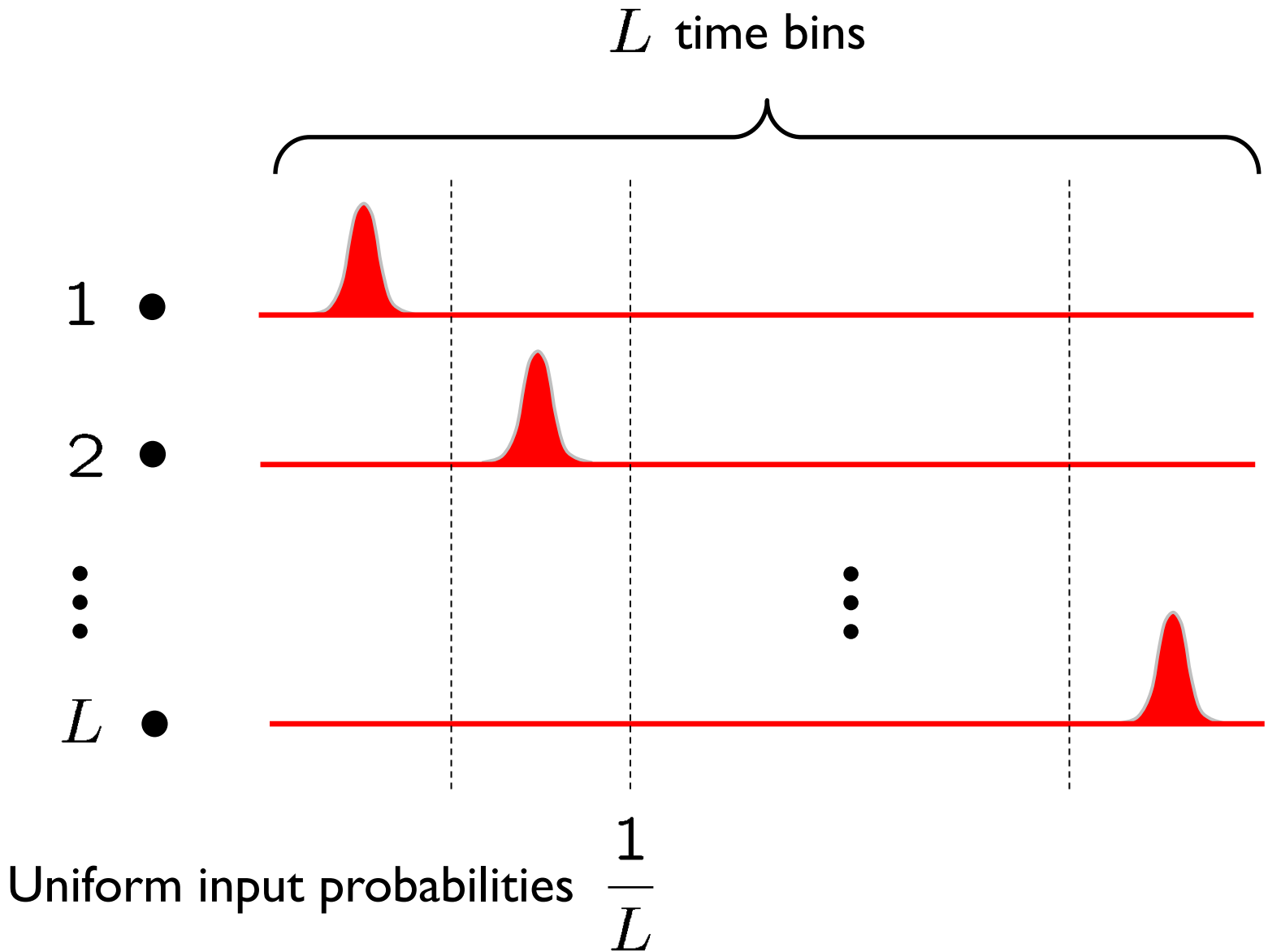
Entropy of individual states:

$$S(\Lambda(\hat{\rho}_n)) > 0$$

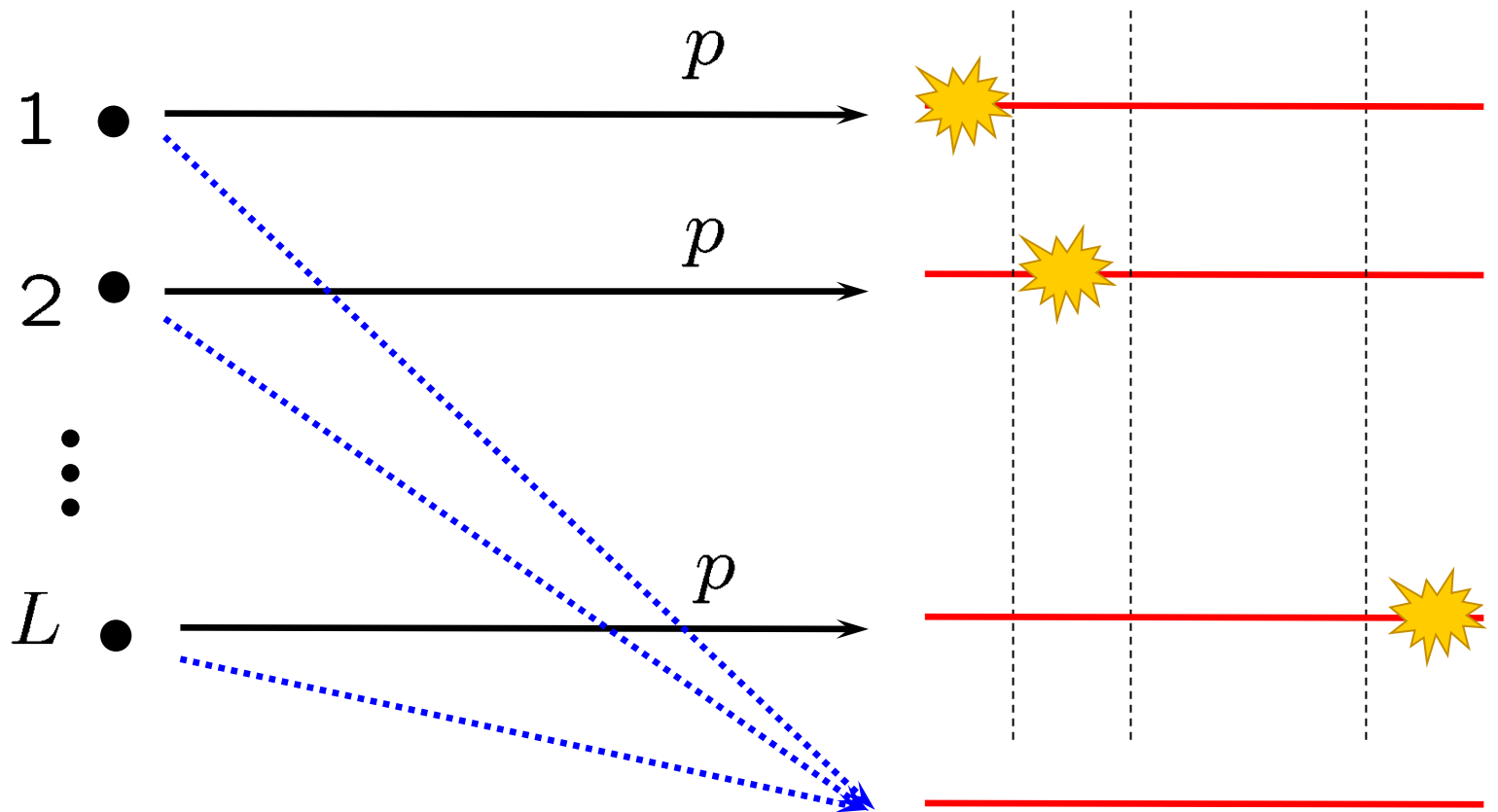
Holevo quantity:

$$\chi(\{p_n, \hat{\rho}_n\}) \approx \eta \bar{n} \log_2 \frac{1}{\bar{n}} + O(\bar{n}^2)$$

PPM: Pulse Position Modulation



Direct detection



Erasure probability

$$\text{.....} \rightarrow 1 - p = \langle : \exp(-\eta \hat{N}) : \rangle$$

Maximum transmission rate

Mutual information per time bin

$$R = \frac{p}{L} \log_2 L$$

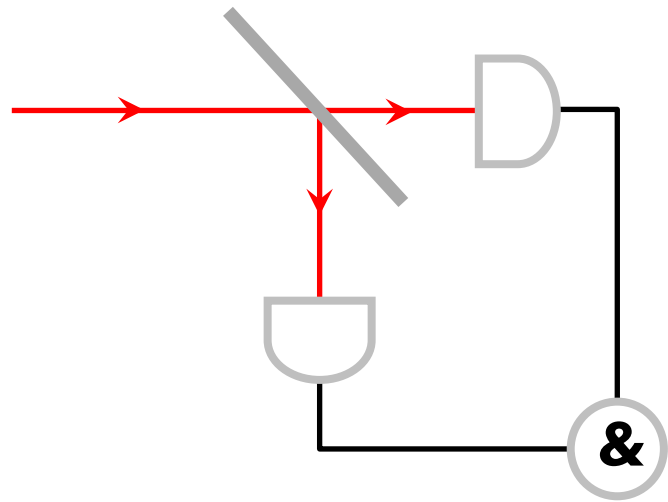
Energy constraint

$$\langle \hat{N} \rangle = L\bar{n}$$

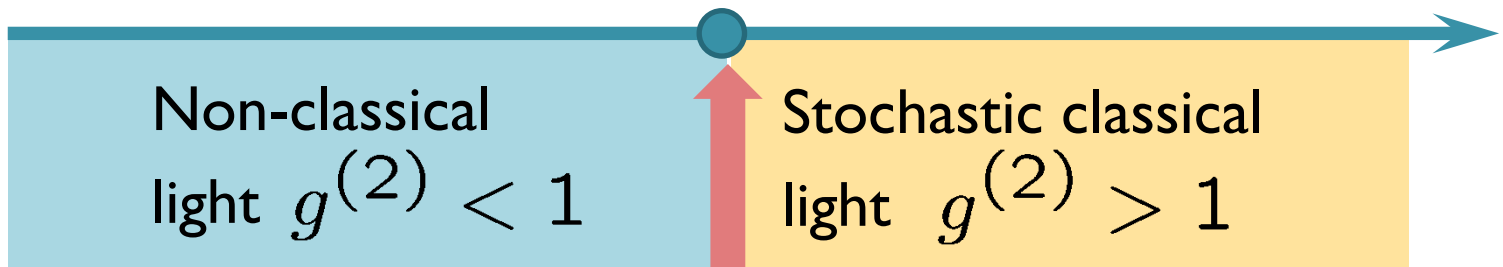
Weak output pulse regime, $\eta \langle \hat{N} \rangle \ll 1$

$$\begin{aligned} p &\approx \eta \langle \hat{N} \rangle - \frac{1}{2} \eta^2 \langle : \hat{N}^2 : \rangle \\ &= L\eta\bar{n} - \frac{g^{(2)}}{2} (L\eta\bar{n})^2 \end{aligned}$$

Intensity correlations



$$g^{(2)} = \frac{\langle : \hat{N}^2 : \rangle}{\langle \hat{N} \rangle^2}$$



Poissonian statistics $g^{(2)} = 1$

Optimisation

$$R = \eta \bar{n} \left(1 - \frac{g^{(2)}}{2} L \eta \bar{n} \right) \log_2 L$$

For sequence length $L \gg 1$ treated as a continuous and unconstrained parameter:

$$L_{\text{opt}} = \frac{2}{g^{(2)} \eta \bar{n}} \left[W \left(\frac{2e}{g^{(2)} \eta \bar{n}} \right) \right]^{-1}$$

Lambert function $W(x) \approx \ln x$ for $x \gg 1$

Hierarchy in the low-power regime:

$$\eta \bar{n} \ll L_{\text{opt}} \eta \bar{n} \approx \frac{2}{g^{(2)}} \left(\ln \frac{2e}{g^{(2)} \eta \bar{n}} \right)^{-1} \ll 1$$

Efficiency

M. Jarzyna, P. Kuszaj, K. Banaszek, Opt. Express **23**, 3170 (2015)

Optimal transmission rate

$$R = \eta \bar{n} \Pi \left(g^{(2)} \eta \bar{n} \right)$$

Photon information efficiency (PIE) for Poissonian statistics:

$$\Pi(\eta \bar{n}) = \frac{1}{\ln 2} \left\{ W \left(\frac{2e}{\eta \bar{n}} \right) - 2 + \left[W \left(\frac{2e}{\eta \bar{n}} \right) \right]^{-1} \right\}$$

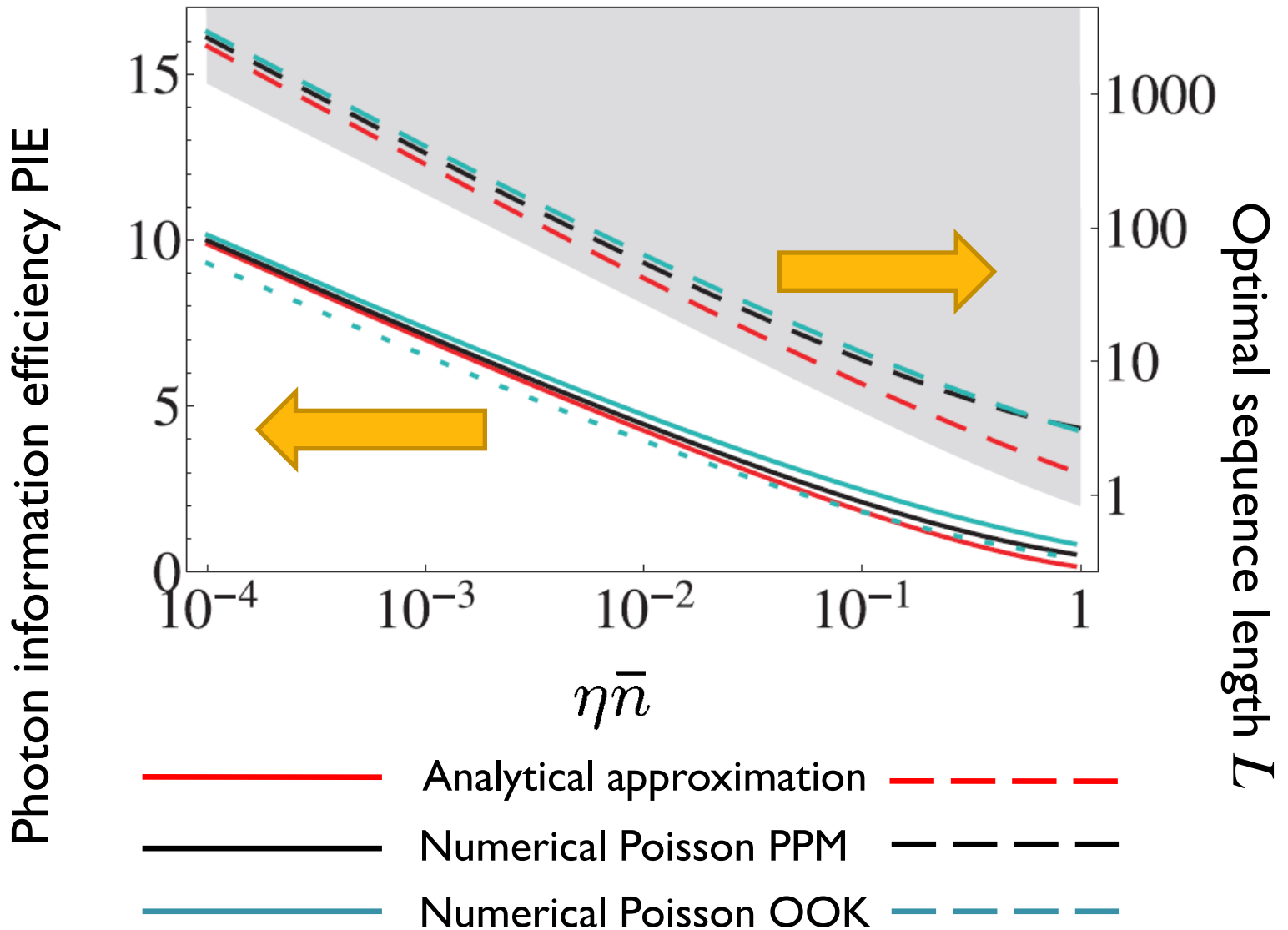
Asymptotics

$$\Pi(\eta \bar{n}) = \log_2 \frac{1}{\eta \bar{n}} - \log_2 \ln \frac{1}{\eta \bar{n}} + O(1)$$

Channel capacity:

$$\frac{1}{\eta \bar{n}} C = \log_2 \frac{1}{\eta \bar{n}} + \frac{1}{\ln 2} + O(\eta \bar{n})$$

Comparison



Nonclassical light

$$R = \eta \bar{n} \Pi \left(g^{(2)} \eta \bar{n} \right)$$

For nonclassical light optimization over the pulse photon number is *constrained*:

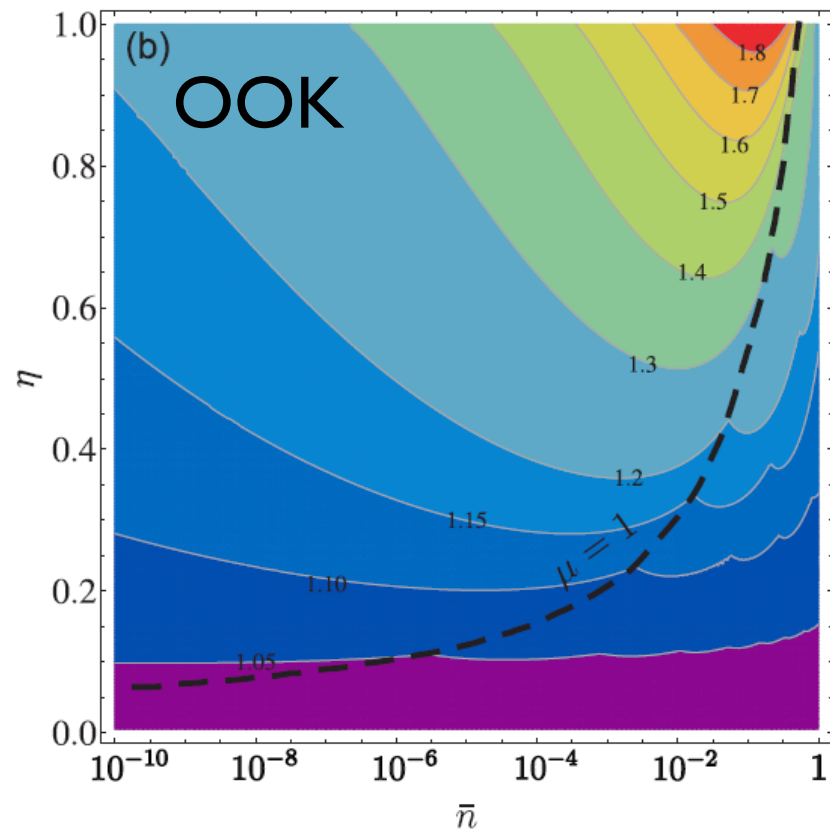
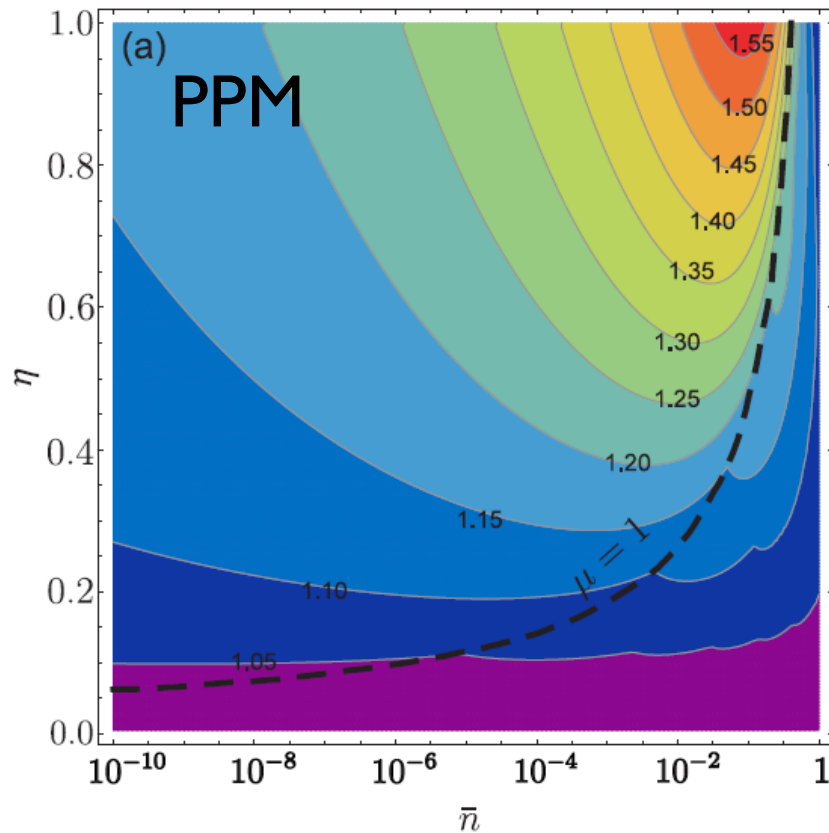
$$\langle \hat{N}^2 \rangle \geq \langle \hat{N} \rangle^2 \quad \longrightarrow \quad \langle \hat{N} \rangle \leq \frac{1}{1 - g^{(2)}}$$

When $\eta > 2 / \ln \frac{1}{\bar{n}}$, single photons are optimal

$$R_{\text{PPM}} = \eta \bar{n} \log_2 \frac{1}{\bar{n}}$$

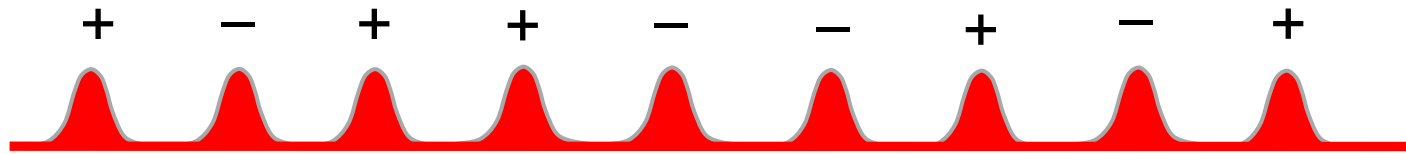
Nonclassical enhancement

$R_{\text{Fock}}/R_{\text{Poisson}}$:

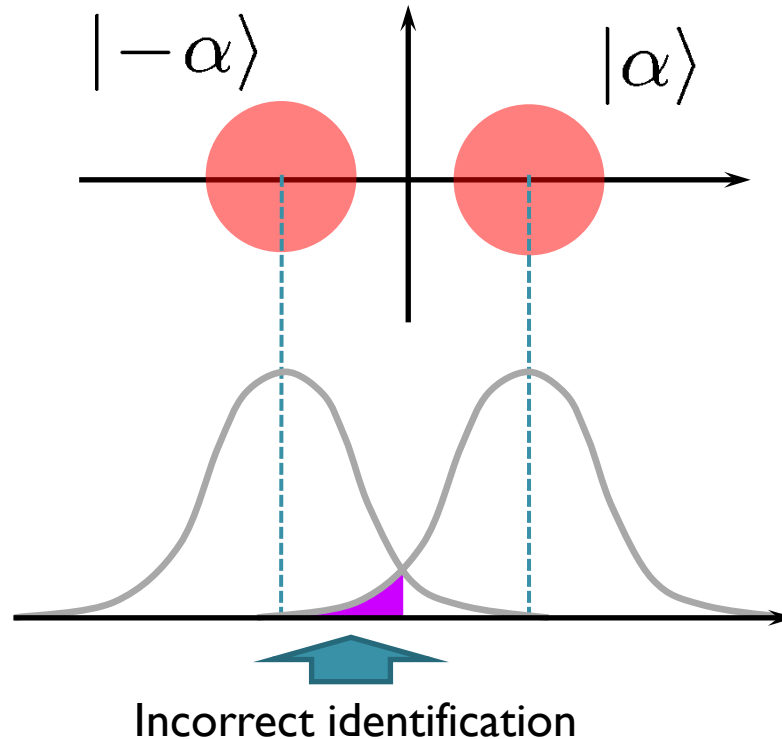


Binary phase shift keying (BPSK)

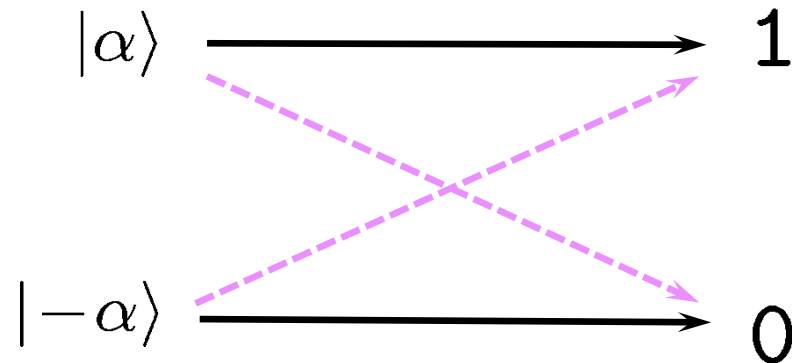
$$\alpha = \sqrt{n}$$



Homodyne detection



Binary symmetric channel



Capacity: $C = 1 - H(e)$

$$H(e) = -e \log_2 e - (1 - e) \log_2 (1 - e)$$

Error probability for homodyne detection

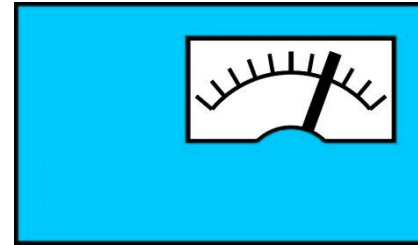
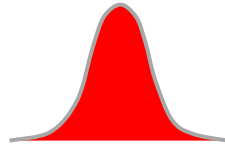
$$e_{\text{hom}} = \frac{1}{2} [1 - \text{erf}(\sqrt{2}\alpha)]$$

Transmission rate

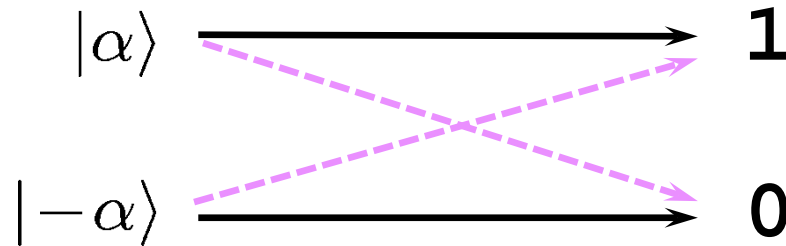
$$R_{\text{hom}} = 1 - H(e_{\text{hom}}) \approx \frac{4\bar{n}}{\pi \ln 2} \quad \text{for } \bar{n} \ll 1$$

Minimum-error measurement

$|\alpha\rangle$ or $|\!-\alpha\rangle$



$$\langle\!-\alpha|\alpha\rangle = e^{-2\bar{n}}$$



Helstrom limit

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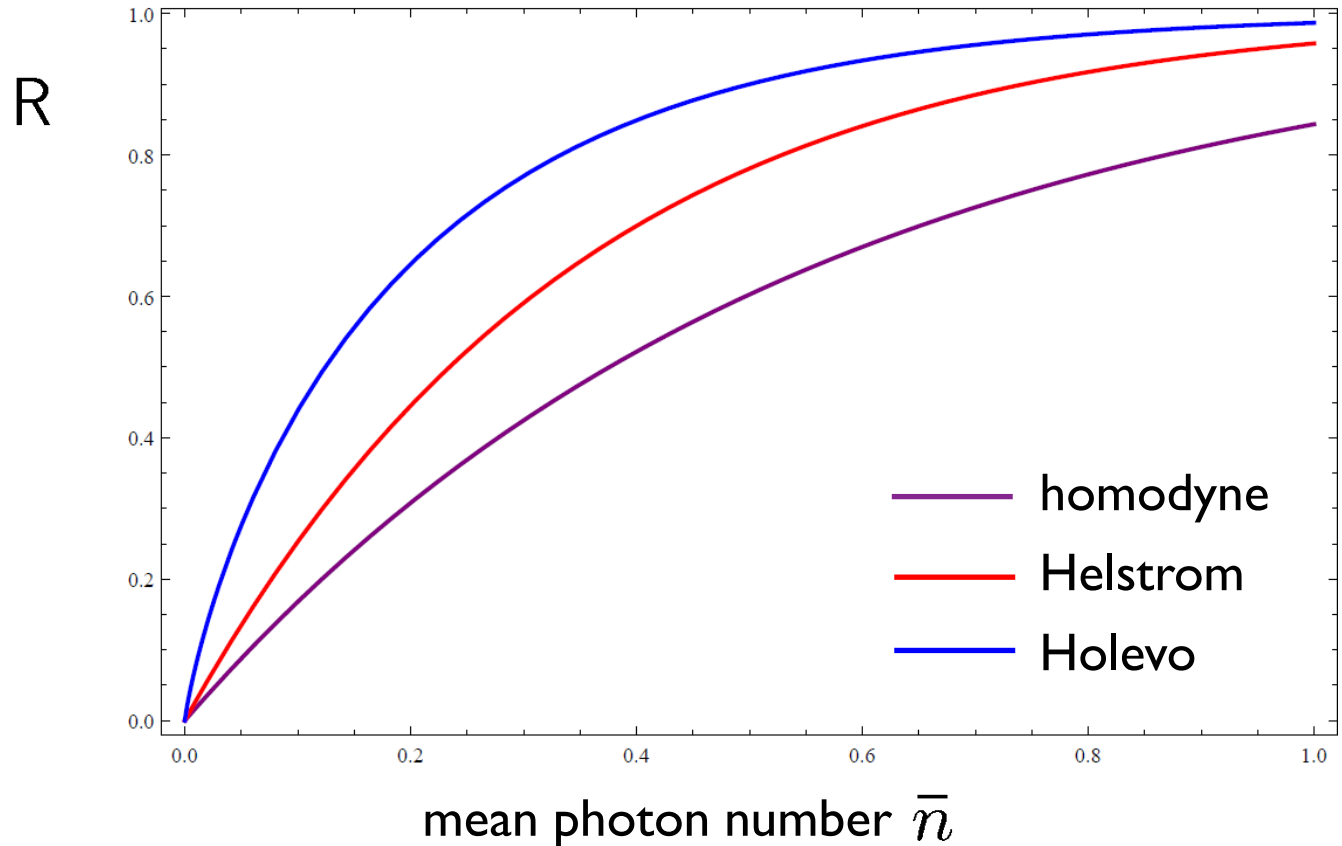
$$e_{\text{Hel}} = \frac{1}{2} \left(1 - \sqrt{1 - |\langle\!-\alpha|\alpha\rangle|^2} \right)$$

Maximum transmission rate *for individual detection*:

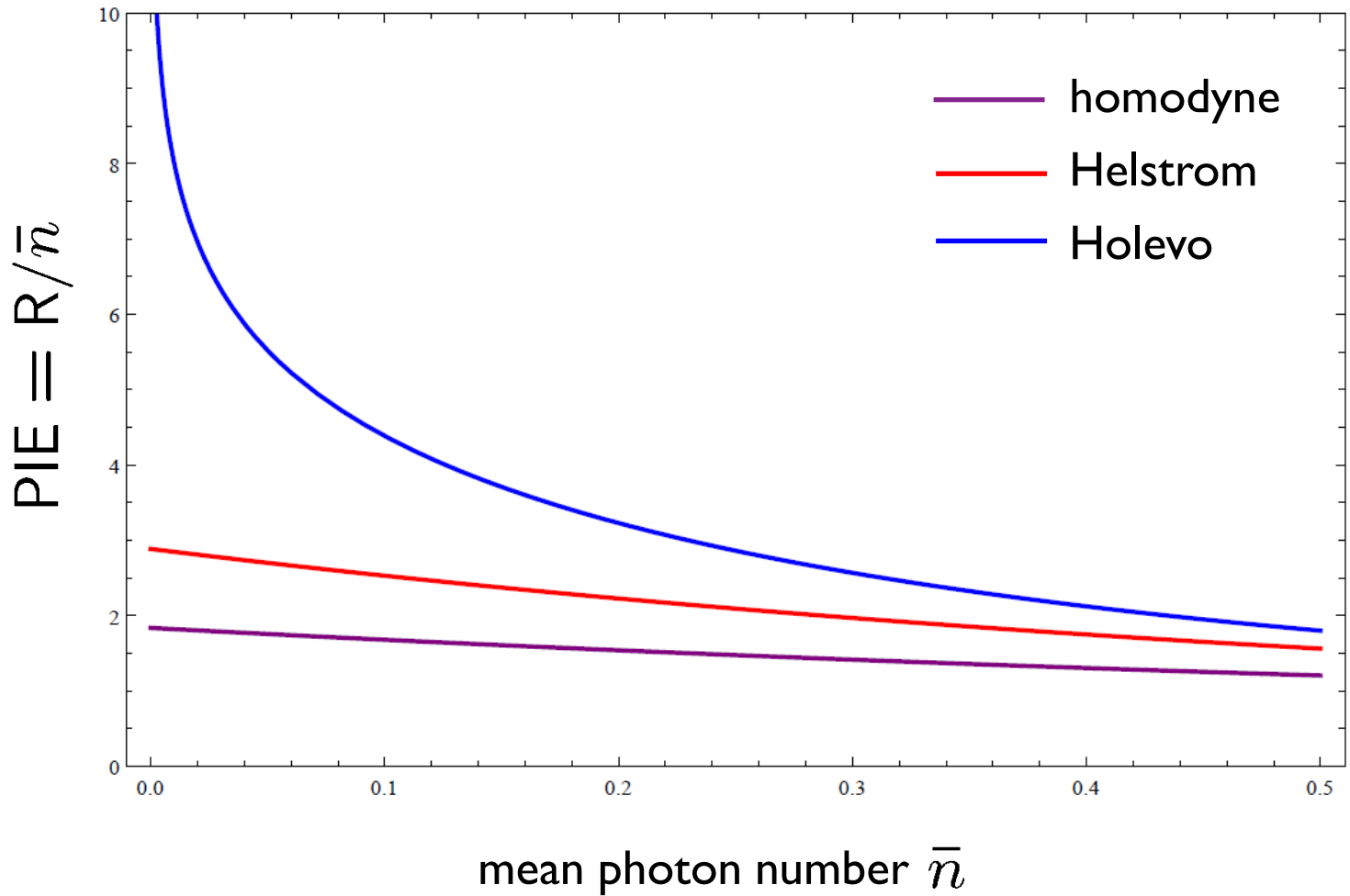
$$R_{\text{Hel}} = 1 - H(e_{\text{Hel}}) \approx \frac{2\bar{n}}{\ln 2}$$

Holevo quantity

$$\chi = S\left(\frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|-\alpha\rangle\langle-\alpha|\right)$$
$$\approx H(\bar{n}) = \bar{n} \log_2 \frac{1}{\bar{n}} + \frac{\bar{n}}{\ln 2} + O(\bar{n}^2)$$

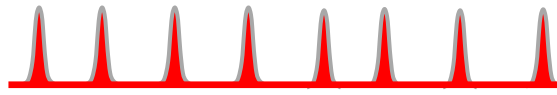


Photon information efficiency



Hadamard words

S. Guha, Phys. Rev. Lett. **106**, 240502 (2011)

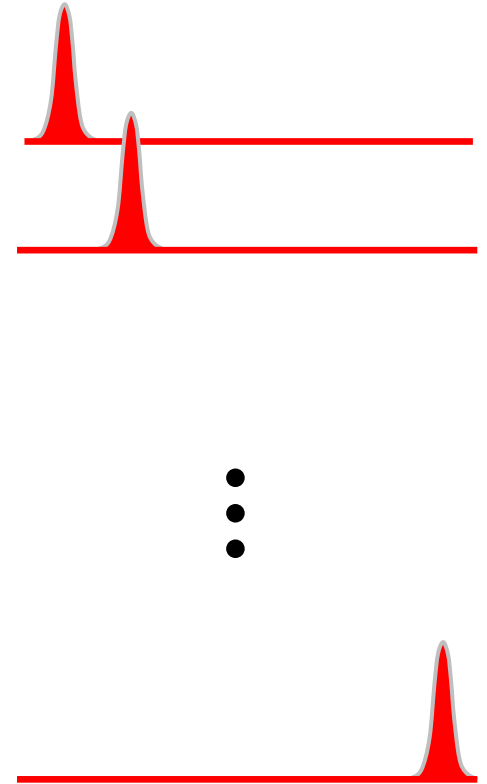


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+	+	-	-	-	-	+	+
+	-	-	+	-	+	+	-

BPSK

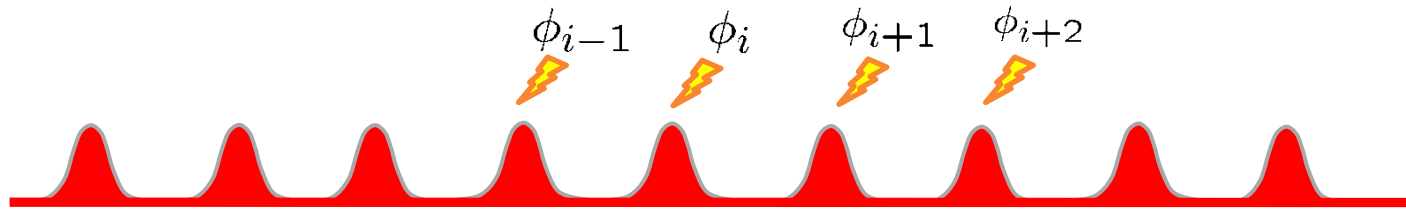


Hadamard transform



PPM

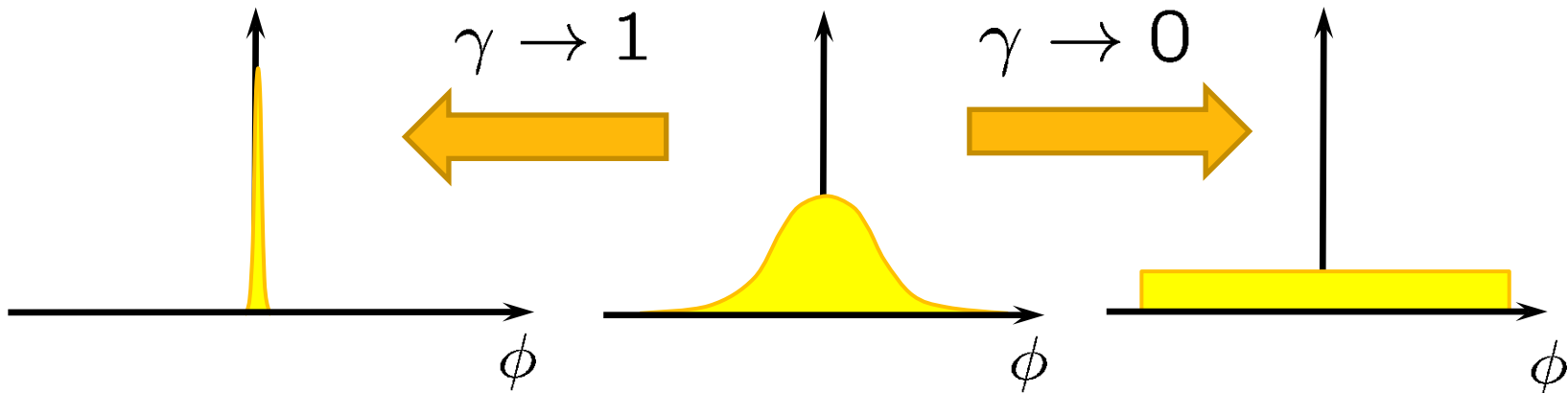
Dephasing



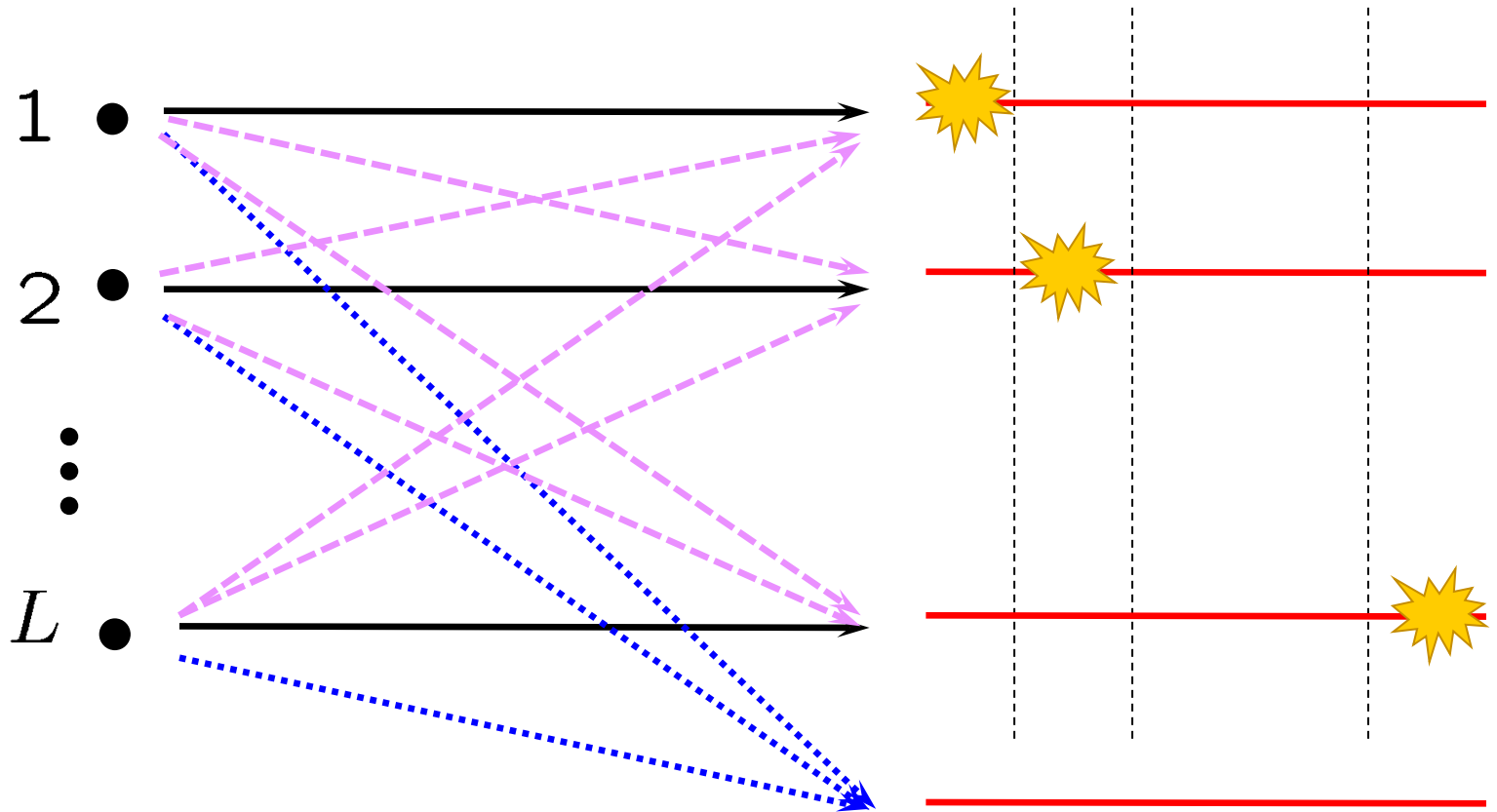
Phase distribution

$$p(\phi) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\phi^2/2\sigma^2}$$

Dephasing parameter $\gamma = e^{-\sigma^2}$



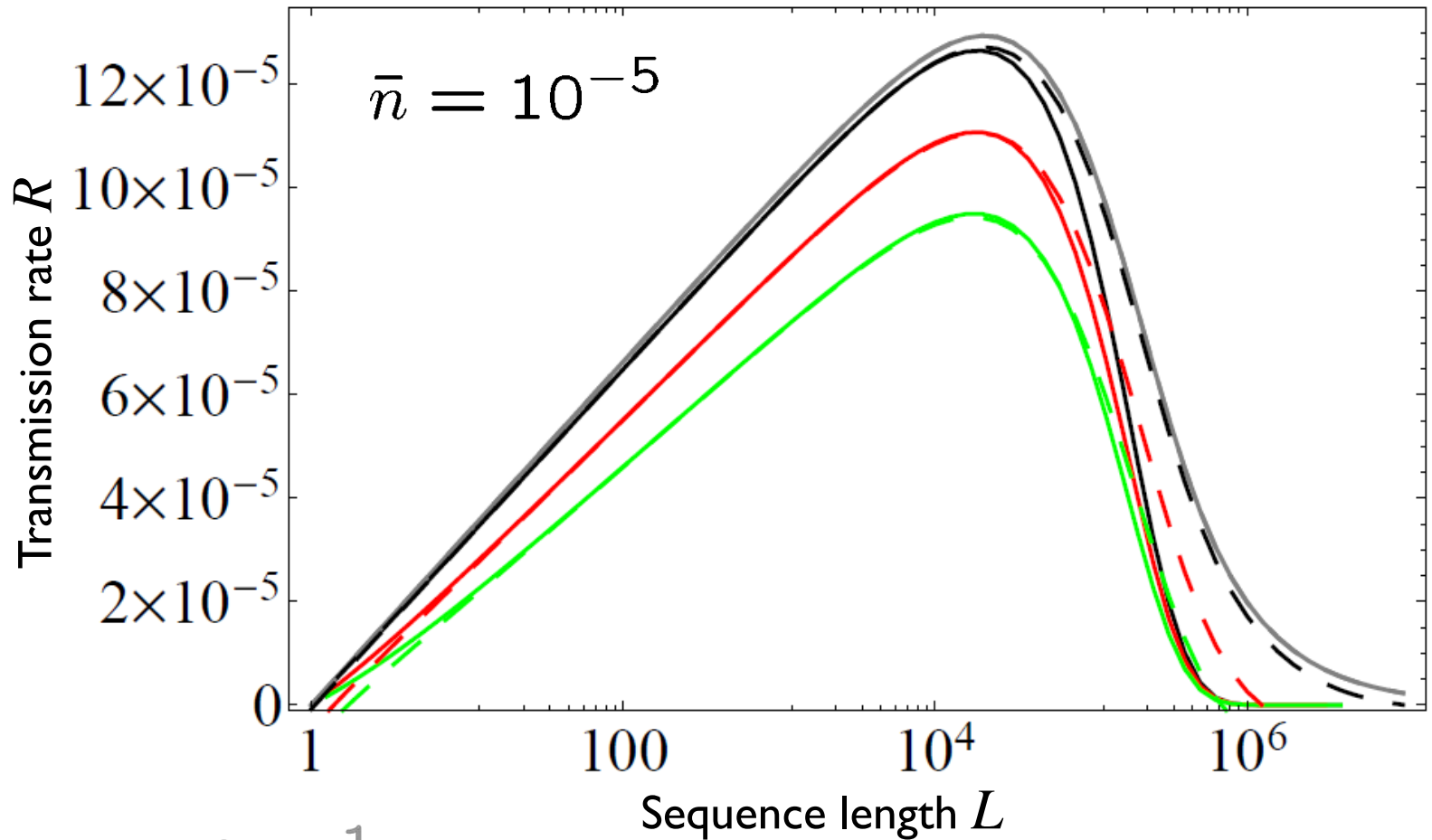
PPM with uniform noise



$$\longrightarrow p \geq \tilde{p} = e^{-(L-1)\bar{n}(1-\gamma)} - e^{-L\bar{n}}$$

$$\dashrightarrow q \leq \tilde{q} = \bar{n}(1 - \gamma)$$

Optimal sequence length



— $\gamma = 1$

Numerical with p, q expanded up to \bar{n}^2 :

— $\gamma = 0.99$ — $\gamma = 0.9$ — $\gamma = 0.8$

Transmission rate

$$R \approx \frac{p}{L} \log_2 L - \bar{n}H(\gamma)$$

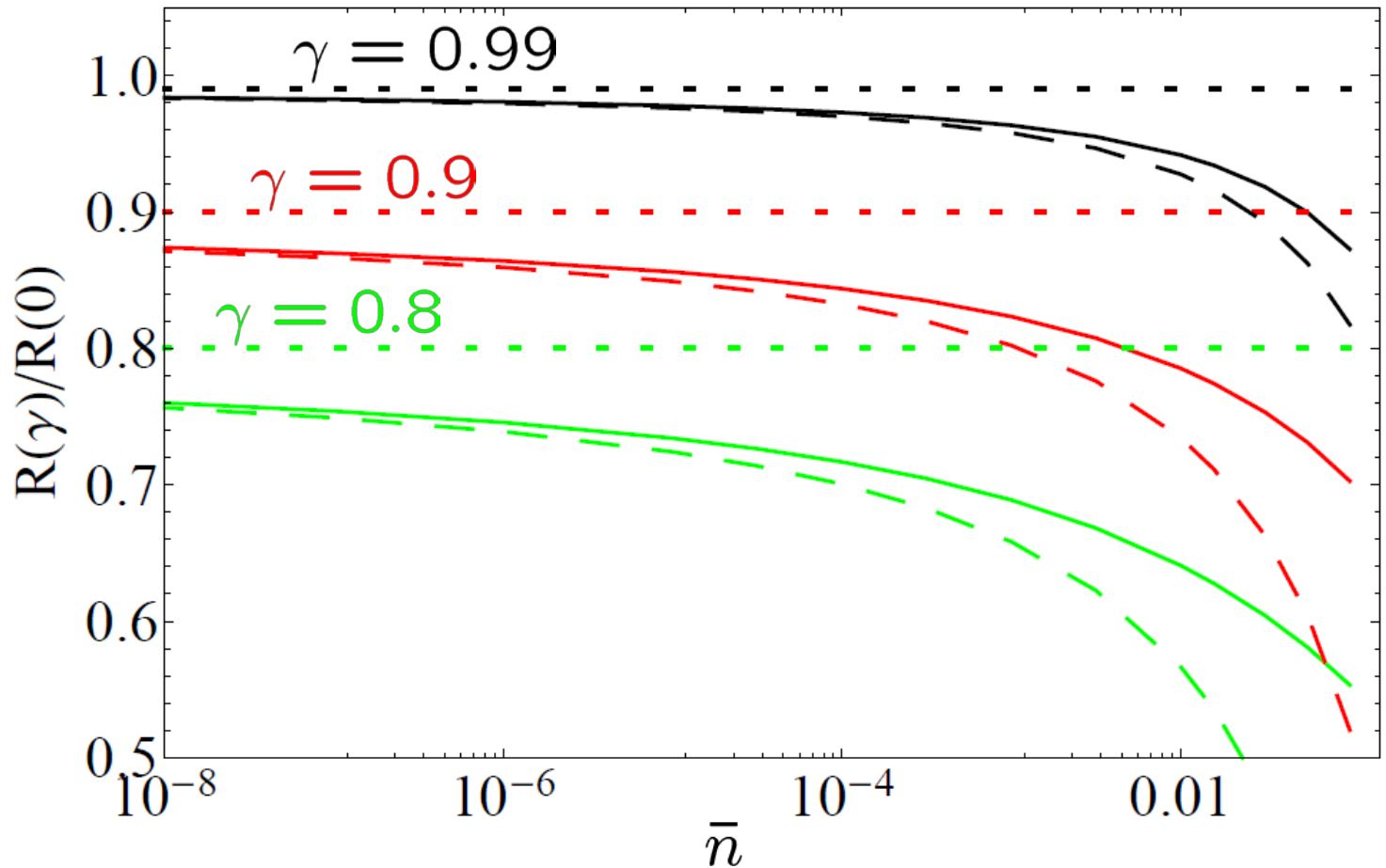
Expanding p up to $(L\bar{n})^2$:

$$p \approx \underbrace{\gamma L \bar{n}}_{\text{Effective } \langle \hat{N} \rangle} - \frac{1}{2} \underbrace{(2\gamma^{-1} - 1)}_{\text{Effective } g^{(2)} > 1} (\gamma L \bar{n})^2$$

Closed analytical expression:

$$\tilde{R} = \gamma \bar{n} \Pi((2 - \gamma)\bar{n}) - \bar{n}H(\gamma).$$

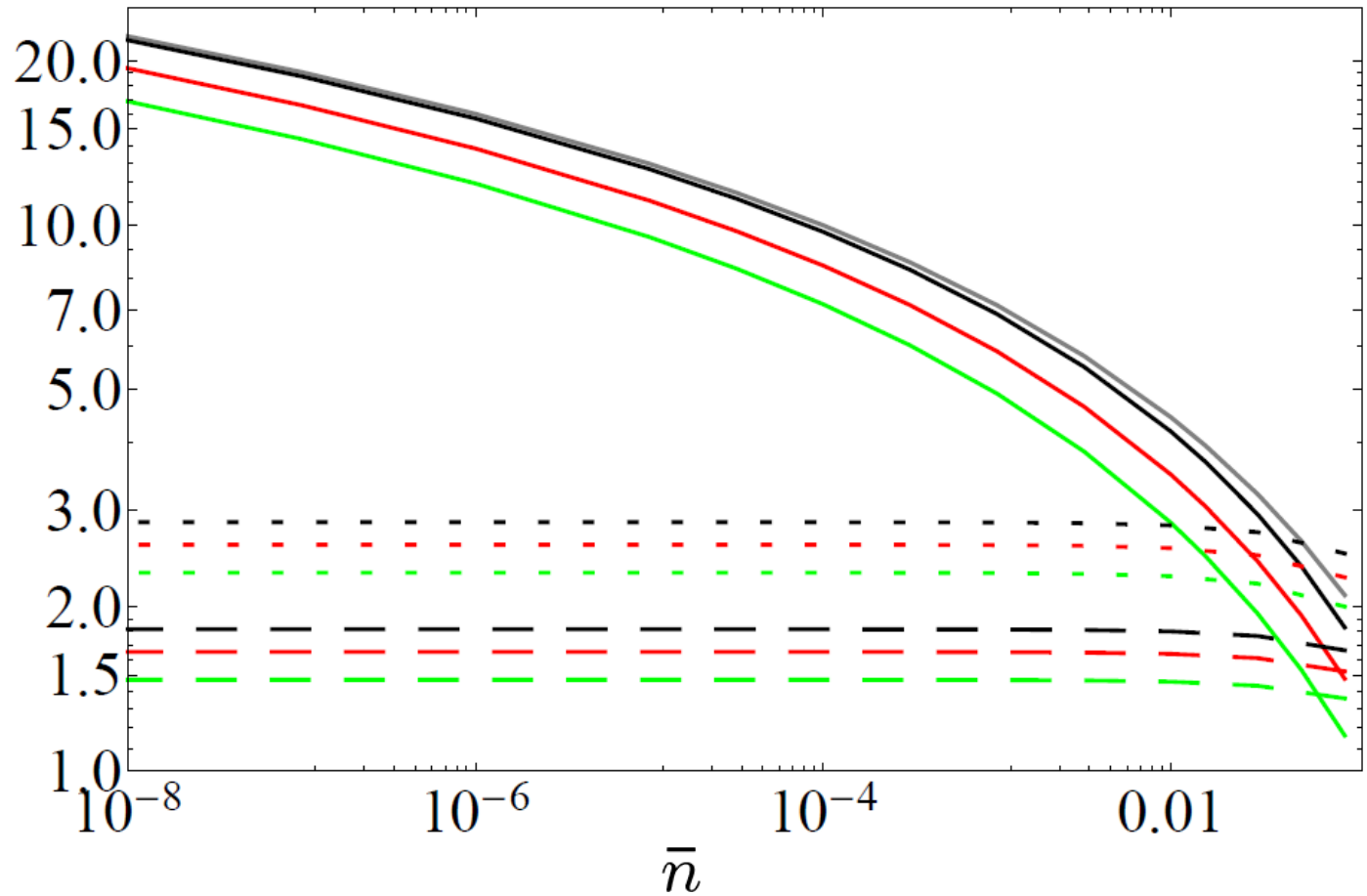
Approximation



- Numerical with p, q expanded up to \bar{n}^2
- - - Closed analytical expression for \tilde{R}

Asymptotics: $R \approx \gamma \bar{n} \log_2 \frac{1}{\bar{n}}$

Photon information efficiency



Homodyne

Helstrom

Collective

$\gamma = 0.99$

.....

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$\gamma = 0.9$

.....

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$\gamma = 0.8$

.....

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Conclusions

- Approximate analytical expression for maximum PPM / OOK transmission rates in the low-power regime
- Approaching channel capacity in the leading order of the average power
- Quantified effects of excess noise in pulse photon statistics
- Non-classical enhancement in the low-loss regime
- Phase noise in collective BPSK
- Open questions:
 - Robustness to dark counts, amplification noise, etc.
 - Resources needed to saturate channel capacity (coherent / collective detection)